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–Brief Paper–

ADAPTIVE MITTAG-LEFFLER STABILIZATION OF A CLASS OF FRACTIONAL ORDER UNCERTAIN NONLINEAR SYSTEMS

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ABSTRACT

This paper deals with the stabilization of a class of commensurate fractional order uncertain nonlinear systems. The fractional order system concerned is of the strict-feedback form with uncertain nonlinearity. An adaptive control scheme combined with fractional order update laws is proposed by extending classical backstepping control to fractional order backstepping scheme. The asymptotic stability of the closed-loop system is guaranteed under the construction of fractional Lyapunov functions in the sense of generalized Mittag-Leffler stability. The fractional order nonlinear system investigated can be stabilized asymptotically globally in presence of arbitrary uncertainty. Finally illustrative examples and numerical simulations are performed to verify the effectiveness of the proposed control scheme.

Key Words: Fractional order systems, uncertainty, Lyapunov function, Mittag-Leffler stability, adaptive control, nonlinearity.

I. INTRODUCTION

Fractional order systems (FOSs) have attracted considerable attentions in recent years. Fractional calculus generalized the classical integer order integration and differentiation to arbitrary non-integer order, thus providing a more accurate modeling to describe real world physical phenomena [1,2]. It was found that a variety of physical and biological systems can be well characterized by fractional order differential equations, such as the fractional order (FO) Schrodinger equation in quantum mechanics and FO oscillator in damping vibration [3,4]. Moreover, FOSs have been found by directly generalizing integer order derivatives into the corresponding fractional order ones, in which way chaotic dynamics has been explored, such as FO Chen's system and FO cellular neural network[2].

The control problems of FOSs have been investigated in the control community, due to the demonstrated applications of fractional calculus in many fields of engineering and science. The stability criteria of fractional order linear systems (FOLSs) have been fully studied [5–9]. The stability of linear time invariant incommensurate or commensurate FOLSs can be analyzed through the investigation of the eigenvalues of system matrix [10,11]. Sufficient and necessary conditions on the stability of fractional order interval systems are investigated by a linear matrix inequality (LMI) approach [12] and state feedback stabilization of FOLSs in triangular form is studied[13]. Selected stabilization problems of positive FOLSs have been addressed [14].On the other hand, for fractional order nonlinear systems (FONSs), sufficient conditions on equilibrium asymptotical stability are given for incommensurate or commensurate FO non-autonomous systems[15] and the Lyapunov stability of fractional differential equations is addressed by the frequency distributed fractional integrator model [16]. Inspired by the above stability analysis of FOSs, various control methods are proposed. The variable-order fractional fuzzy proportional–integrail–derivative (PID) controller is proposed to achieve better system performance [17]. FO sliding surfaces and FO sliding mode reaching laws have been proposed to deal with fractional order uncertain chaotic systems [18–20], with application in permanent magnet synchronous motor and secure communication. However, none of the above mentioned works was based on the fractional Lyapunov theory and most of these works only considered nonlinearly parameterized uncertainty without the case of arbitrary uncertainty. Furthermore, there still remain certain traditional control methods that have been applied successfully into integer order systems to be extended and generalized to FOSs, such as the backstepping design.

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It is well known that the Lyapunov direct method has become one of the most important tools to stability analysis and control design of nonlinear systems. The necessary conditions for the stability defined in terms of two measures for FOSs have been provided in a versatile approach by utilizing Lyapunov-like functions [21], however it is hard to involve system differential equations in the fractional derivative of the Lyapunov functions proposed. Other Lyapunov functionals are then proposed [22], but are not simple and only valid for fractional differential equations written as integral equations. The fractional order extension of the Lyapunov direct method is proposed to analyze Lyapunov-like stability, by introducing generalized Mittag-Leffler stability [23,24]. The generalized Mittag-Leffler stability of multivariables FOSs [25] and Lyapunov uniform stability [26] for FOSs are then investigated respectively. Fortunately, some new fractional derivative properties and fractional comparison principles have been developed in recent papers [26,27], which help to find appropriate fractional Lyapunov functions.

Backstepping control design has been applied broadly in stabilizing a general class of integer order nonlinear systems in real applications. Up to now, a great amount of research has been reported for classical nonlinear systems in strict-feedback form or lower triangular systems [28,29], and many control methods have been incorporated into the backstepping design [30,31]. However, to the best of our knowledge, there are few results on the generalizations of backstepping control into FONSs, other than the proposed example [32] and our previous theoretical result [33]. Therefore, it is of great worth to explore more backstepping techniques for FONSs.

In the previous work [33], control of FOSs with unknown parameters has been studied, without the consideration of the more general uncertainties, such as external disturbances and noises. In this paper, we consider a more general case where the FOSs contain uncertain functions, which is more complicated than our previous work.

In our contributions, we investigate the stabilization problem, in terms of generalized Mittag-Leffler stability, of a class of commensurate FONSs in strict-feedback form with arbitrary uncertainty. As far as we know, there have been no results on this issue before. It is should be noted that Mittag-Leffler asymptotic stability implies Lyapunov asymptotic stability. By means of extending backstepping control into FONSs, an adaptive control scheme combined with uncertainty estimates is proposed. A general framework of quadratic Lyapunov candidate functions is constructed for stability analysis and the global asymptotic stabilization of the closed-loop system is guaranteed in sense of fractional Lyapunov stability. The arbitrary uncertainties are approximated by RBF neural networks. In our design, the unknown parameters and unknown upper bounds of the approximation errors are estimated by the proposed fractional order update laws. The uncertain parameters and the approximation errors are assumed to be unknown constants, and to be bounded by unknown upper bounds respectively. The parameters in our design are irrespective of the system uncertainties, and thus can be chosen freely for better regulation of the controlled FONS. Finally, illustrative examples and simulations are presented to demonstrate the validity of our control design.

The rest of the paper is organized as follows. In Section II, mathematical preliminaries, stability criteria of FONSs are introduced. In Section III, the adaptive control design via backstepping and fractional update laws are presented. In Section IV, illustrative examples and the simulations are provided.

II. PRELIMINARIES

Definition 1. Let $f : [a, b] \to R$ and $f \in L^1[a, b]$. The Caputo definition of fractional derivative of order q is defined as

$$D_t^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{q-n+1}} d\tau \tag{1}$$

where q is the derivative order and $\Gamma(\cdot)$ is the Gamma function. $f \in L^1[a, b]$ signifies that f is Lebesgue measurable on the interval $[a, b]$.

In this paper we use the symbol D^q for D_t^q and D for the classical integer differential $D^1 f(t) = df(t)/dt$.

Definition 2. [24] The constant x_0 is an equilibrium point of the Caputo fractional dynamic system $D^q x = f(x, t)$, $x \in \mathbb{R}$, if and only if $f(x_0, t) = 0$. Without loss of generality, let the equilibrium be $x_0 = 0$.

Definition 3. [34] A continuous function $\gamma : [0, t) \to [0, \infty)$ is said to belong to class-K if it is strictly increasing and $\gamma(0) = 0$.

Lemma 1. [34] Let $V : D \to \mathbb{R}$ be a continuous positive definite function defined on a domain $D \subset \mathbb{R}^n$ that contains the origin. Let $B_d \subset D$ for some $d > 0$. Then there exist class-K functions γ_1 and γ_2 defined on $[0, d]$, such that

$$\gamma_1(\|x\|) \le V(x) \le \gamma_2(\|x\|) \tag{2}$$

for all $x \in B_d$. If $D = \mathbb{R}^n$, the functions γ_1 and γ_2 will be defined on $[0, \infty)$.

Stability analysis of fractional order system by means of Lyapunov theory has been investigated in[23–26] and the main result can be demonstrated by the following theorem.

Theorem 1. [24] Let $x_0 = 0$ be the equilibrium point of the fractional order system $D^q x = f(x,t)$, $x \in D \subset \mathbb{R}^n$ where D contains the origin. Assume that a fractional Lyapunov function $V(x,t) : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ is a continuous differential function and locally Lipschitz with respect to x, and there exist class-K functions $\gamma_i (i = 1, 2, 3)$ such that

$$\begin{aligned} \gamma_1(\|x\|) \le V(x,t) \le \gamma_2(\|x\|) \\ D^q V(x,t) \le -\gamma_3(\|x\|) \end{aligned} \tag{3}$$

then $x(t) = 0$ is asymptotically stable. Moreover, if the conditions hold globally on $D = \mathbb{R}^n$, then is $x(t) = 0$ globally asymptotically stable.

Remark 1. Theorem 1 extends the Lyapunov direct method into FOSs and it is called the generalized Mittag-Leffler stability theorem. It should be noted that Mittag-Leffler asymptotic stability implies the Lyapunov asymptotic stability[24].

Since Theorem 1 describes the system stability in the form of fractional derivative of the Lyapunov function, the following properties are useful to construct the fractional Lyapunov function.

Lemma 2. [27] Let $x(t) \in \mathbb{R}$ be a real-valued continuous and derivable vector function. Then, for any time $t \ge 0$, it is always hold that

$$\frac{1}{2} D^q x^2(t) \le x(t) D^q x(t) \tag{4}$$

where $q \in (0,1)$.

Lemma 3. [27] Let $x(t) = [x_1(t), \dots, x_n(t)] \in \mathbb{R}^n$ be a real-valued continuous and derivable vector function. Then, for any time $t \ge 0$, it always holds that

$$\frac{1}{2} D^q x(t)^\top P x(t) \le x(t)^\top P D^q x(t) \tag{5}$$

where $q \in (0,1)$ and $P = diag[p_1, \dots, p_n] > 0$.

Remark 2. The differential equations of the FOSs are related with the fractional Lyapunov function in quadratic form by the inequalities in Lemma 2 and Lemma 3. Thus these two lemmas help to construct appropriate Lyapunov functions for FONS, as will be seen in our proof in the next section.

III. MAIN RESULTS

We consider the fractional order nonlinear systems in strict-feedback form with arbitrary uncertainty described by

$$\begin{aligned} D^q x_1(t) &= x_2 + f_1(x_1) \\ D^q x_i(t) &= x_{i+1} + f_i(x_1, x_2, \dots, x_i) \\ D^q x_n(t) &= u + f_n(x) \\ y &= x_1 \end{aligned} \tag{6}$$

where $i = 2, \dots, n-1$; the system state $x \in \mathbb{R}^n$, y is the output and $q \in (0,1)$ is the order of the fractional derivative; $f_i (i = 1, \dots, n)$ are unknown smooth vector valued functions and u is the control.

Assumption 1. The unknown functions f_i can be expressed as

$$f_i(x_1, x_2, \dots, x_i) = \varphi_i^\top (x_1, x_2, \dots, x_i)\theta_i + d_i \tag{7}$$

where $\varphi_i^\top$ are continuous vector valued functions with $\varphi_i^\top(0) = 0$; $\theta_i \in \mathbb{R}^{m_i}$ are the unknown constant parameters; d_i are bounded approximation errors with unknown upper bounds $\delta_i \ge |d_i|$.

Remark 3. Assumption 1 can be satisfied by the approach of function approximation, such as neural networks and Fourier series expansion. The stabilization of the integer order form of the system (6) has been fully studied [28–31], however, limited results on its fractional order form have been reported.

Under the Assumption 1, the considered system (6) is expressed as

$$\begin{aligned} D^q x_1(t) &= x_2 + \varphi_1^\top (x_1)\, \theta_1 + d_1 \\ D^q x_i(t) &= x_{i+1} + \varphi_i^\top (x_1, \dots, x_i)\, \theta_i + d_i \\ D^q x_n(t) &= u + \varphi_n^\top(x)\theta_n + d_n \end{aligned} \tag{8}$$

Theorem 2. The fractional order system (8) with uncertainties can be stabilized globally asymptotically by the adaptive feedback control

$$u = -\left[z_{n-1} + k_n z_n + \varphi_n^\top(x)\hat{\theta}_n + sign\left(z_n\right)\hat{\delta}_n - D^q \beta_{n-1}\right] \tag{9}$$

where k_i are positive constants and

$$\begin{aligned} \beta_i = &-\left[z_{i-1} + k_i z_i + \varphi_i^\top (x_1, \dots, x_i)\, \hat{\theta}_i \right. \\ &\left. + sign(z_i)\hat{\delta}_i - D^q \beta_{i-1}\right], i = 2, \dots, n-1 \end{aligned} \tag{10}$$

and $\beta_1 = -\left[k_1 z_1 + \varphi_1^\top(x_1)\hat{\theta}_1 + sign(z_1)\hat{\delta}_1\right]$, with update laws

$$D^q\hat{\theta}_i = \Gamma_i \varphi_i(x_1, \dots, x_i) z_i \tag{11}$$

and

$$D^q\hat{\delta}_i = r_i \left|z_i\right| \tag{12}$$

where $\hat{\theta}_i$ is the estimate of the unknown parameter θ_i and $\hat{\delta}_i$ is the estimate of the unknown upper bound δ_i in Assumption 1; $\Gamma_i = diag[p_{i1}, \dots, p_{im_i}] > 0$ and $r_i > 0$ are the gains of the update laws respectively.

Proof. We proceed step by step.

Step 1. Let $z_1 = x_1$ and define error variable $z_2 = x_2 - \beta_1(z_1, \hat{\theta}_1, \hat{\delta}_1)$, we obtain

$$D^q z_1(t) = z_2 + \beta_1(z_1, \hat{\theta}_1, \hat{\delta}_1) + \varphi_1^\top(x_1)\theta_1 + d_1 \tag{13}$$

Choose Lyapunov candidate function

$$V_1(z_1, \tilde{\theta}_1, \tilde{\delta}_1) = \frac{1}{2}z_1^2 + \frac{1}{2}\tilde{\theta}_1^\top\Gamma_1^{-1}\tilde{\theta}_1 + \frac{1}{2r_1}\tilde{\delta}_1^2 \tag{14}$$

where $\tilde{\theta}_i = \theta_i - \hat{\theta}_i$ is the parameter estimate error and $\tilde{\delta}_i = \delta_i - \hat{\delta}_i$ is the upper bound estimate error, by applying Caputo derivative, and according to Lemma 2 and Lemma 3, we obtain

$$D^q V_1 \le z_1 D^q z_1 + \tilde{\theta}_1^\top\Gamma_1^{-1}D^q\tilde{\theta}_1 + \frac{1}{r_1}\tilde{\delta}_1 D^q\tilde{\delta}_1 \tag{15}$$

By equation (13), we obtain

$$\begin{aligned} D^q V_1 &\le z_1\left(z_2 + \beta_1(z_1, \hat{\theta}_1, \hat{\delta}_1) + \varphi_1^{\top\theta_1} + d_1\right) \\ &\quad - \tilde{\theta}_1^\top\Gamma_1^{-1}D^q\hat{\theta}_1 - \frac{1}{r_1}\tilde{\delta}_1 D^q\hat{\delta}_1 \\ &= z_1\left(z_2 + \beta_1 + \varphi_1^{\top\theta_1}\right) + z_1 d_1 \\ &\quad + \tilde{\theta}_1^\top\left(\varphi_1 z_1 - \Gamma_1^{-1}D^q\hat{\theta}_1\right) - \frac{1}{r_1}\tilde{\delta}_1 D^q\hat{\delta}_1 \end{aligned} \tag{16}$$

Choose the first stabilizing function $\beta_1 = -\left[k_1 z_1 + \varphi_1^\top(x_1)\hat{\theta}_1 + sign(z_1)\hat{\delta}_1\right]$ and note that $z_1 d_1 \le \left|z_1 d_1\right| \le \left|z_1\right|\left|d_1\right| \le \left|z_1\right|\delta_1 = \left|z_1\right|(\tilde{\delta}_1 + \hat{\delta}_1)$, we obtain

$$\begin{aligned} D^q V_1 &\le -k_1 z_1^2 + z_1 z_2 + \tilde{\theta}_1^\top\left(\varphi_1 z_1 - \Gamma_1^{-1}D^q\hat{\theta}_1\right) \\ &\quad + \tilde{\delta}_1\left(\left|z_1\right| - \frac{1}{r_1}D^q\hat{\delta}_1\right) + \left|z_1\right|\hat{\delta}_1 - \left|z_1\right|\hat{\delta}_1 \end{aligned} \tag{17}$$

Substituting update laws (11) and (12) gives

$$D^q V_1 \le -k_1 z_1^2 + z_1 z_2 \tag{18}$$

Step 2. Defining error variable $z_3 = x_3 - \beta_2$, we obtain

$$D^q z_2(t) = z_3 + \beta_2 + \varphi_2^{\top\theta_2} + d_2 - D^q\beta_1 \tag{19}$$

Choose the second Lyapunov function $V_2 = V_1 + \frac{1}{2}z_2^2 + \frac{1}{2}\tilde{\theta}_2^\top\Gamma_2^{-1}\tilde{\theta}_2 + \frac{1}{2r_2}\tilde{\delta}_2^2$, we obtain

$$\begin{aligned} D^q V_2 &\le -k_1 z_1^2 + z_2\left(z_3 + \beta_2 + \varphi_2^{\top\theta_2} + d_2 - D^q\beta_1\right) \\ &\quad - \tilde{\theta}_2^\top\Gamma_2^{-1}D^q\hat{\theta}_2 - \frac{1}{r_2}\tilde{\delta}_2 D^q\hat{\delta}_2 + z_1 z_2 \end{aligned} \tag{20}$$

Choosing the second stabilizing function $\beta_2 = -\left[k_2 z_2 + z_1 + \varphi_2^{\top\hat{\theta}_2} + sign(z_2)\hat{\delta}_2 - D^q\beta_1\right]$, by update laws (11) and (12), we obtain

$$D^q V_2 \le -k_1 z_1^2 - k_2 z_2^2 + z_2 z_3 \tag{21}$$

By the same procedure presented above, define $z_{i+1} = x_{i+1} - \beta_i$ with stabilizing functions (10) and update laws (11) and (12). In each step, choose Lyapunov candidate function $V_i = V_{i-1} + \frac{1}{2}z_i^2 + \frac{1}{2}\tilde{\theta}_i^\top\Gamma_i^{-1}\tilde{\theta}_i + \frac{1}{2r_i}\tilde{\delta}_i^2$, we obtain

$$D^q V_{n-1} \le -\sum_{i=1}^{n-1} k_i z_i^2 + z_{n-1}z_n \tag{22}$$

Step *n*. Choose Lyapunov candidate function

$$V_n = V_{n-1} + \frac{1}{2}z_n^2 + \frac{1}{2}\tilde{\theta}_n^\top\Gamma_n^{-1}\tilde{\theta}_n + \frac{1}{2r_n}\tilde{\delta}_n^2 \tag{23}$$

and note that $z_n = x_n - \beta_{n-1}$, we obtain

$$\begin{aligned} D^q V_n &\le -\sum_{i=1}^{n-1} k_i z_i^2 + z_n\left(\varphi_n^{\top\theta_n} + d_n + u - D^q\beta_{n-1}\right) \\ &\quad + z_{n-1}z_n - \tilde{\theta}_n^\top\Gamma_n^{-1}D^q\hat{\theta}_n - \frac{1}{r_n}\tilde{\delta}_n D^q\hat{\delta}_n \\ &= -\sum_{i=1}^{n-1} k_i z_i^2 + z_n\left(z_{n-1} + \varphi_n^{\top\hat{\theta}_n} + u - D^q\beta_{n-1}\right) \\ &\quad + z_n d_n + \tilde{\theta}_n^\top\left(\varphi_n z_n - \Gamma_n^{-1}D^q\hat{\theta}_n\right) - \frac{1}{r_n}\tilde{\delta}_n D^q\hat{\delta}_n \end{aligned} \tag{24}$$

Substitute the adaptive control (9), and note that $z_n d_n \le \left|z_n\right|(\tilde{\delta}_n + \hat{\delta}_n)$, we obtain

$$\begin{aligned} D^q V_n &\le -\sum_{i=1}^{n} k_i z_i^2 + \tilde{\theta}_n^\top\left(\varphi_n z_n - \Gamma_n^{-1}D^q\hat{\theta}_n\right) \\ &\quad + \tilde{\delta}_n\left(\left|z_n\right| - \frac{1}{r_n}D^q\hat{\delta}_n\right) + \left|z_n\right|\hat{\delta}_n - \left|z_n\right|\hat{\delta}_n \end{aligned} \tag{25}$$

By update laws (11) and (12), we obtain

$$D^q V_n \leq -\sum_{i=1}^{n} k_i z_i^2 \tag{26}$$

According to Lemma 1, for the Lyapunov candidate function V_n, there exist class-k functions γ_1 and γ_2 such that

$$\gamma_1(\|\eta\|) \leq V_n(\eta) \leq \gamma_2(\|\eta\|) \tag{27}$$

where $\eta = [z_1, \dots, z_n, \tilde{\theta}_1, ...\tilde{\theta}_n, \tilde{\delta}_1, \dots, \tilde{\delta}_n]^\top$.

Unless $z_i = 0$, we have $D^q V_n < 0$, thus by Lemma 1 there exists a class-K function γ_3 such that

$$D^q V_n \leq -\gamma_3(\|\eta\|) \tag{28}$$

According to Theorem 1, the z-system converges to zero globally asymptotically. Since $\lim_{t\to\infty} z_1 = x_1 = 0$ implies $\lim_{t\to\infty} \beta_1 = 0$, and note that $z_2 = x_2 - \beta_1$ and $\lim_{t\to\infty} z_2 = 0$ implies $\lim_{t\to\infty} x_2 = 0$, by induction, we have $\lim_{t\to\infty} x_i = 0$, thus the system (6) can achieve global asymptotical stabilization under the proposed control. This completes the proof.

Remark 4. In the above design procedure, the backstepping control has been extended to fractional order systems. A fractional Lyapunov function has been constructed and the fractional extension of Lyapunov direct method has been applied to obtain the asymptotical stability of the closed-loop system, in sense of Mittag-Leffler asymptotical stability.

Remark 5. In comparison with the case of nonlinearly parameterized uncertainty in [18–20], our design works with arbitrary uncertainty and the control parameters can be chosen freely irrespective of the system uncertainties, thus the proposed adaptive control is valid for more general fractional order systems in real applications. Furthermore, Theorem 2 makes it also possible to achieve the output tracking a reference trajectory y_r asymptotically by defining $z_1 = x_1 - y_r$ instead in step 1, which case will be used to illustrate the approximation of the unknown functions in our design.

For the approximation errors, if there exists a maximum $\delta = \max\{\delta_i, i = 1, \dots, n\}$, it is sufficient to only estimate δ, instead of all δ_i, to guarantee the system stabilization, which is described by the following theorem.

Theorem 3. The fractional order system (8) with uncertainties can be stabilized globally asymptotically by the adaptive feedback control

$$u = -\left[z_n + z_{n-1} + k_n z_n + \varphi_n^\top(x)\hat{\theta}_n + m_n - D^q \beta_{n-1}\right] \tag{29}$$

where $m_i = z_i\hat{\delta}^2 / \left(\left|z_i\right|\hat{\delta} + z_i^2\right), i = 1, \dots, n$ and k_i are positive constants and

$$\begin{aligned}\beta_i = -&\left[z_i + z_{i-1} + k_i z_i + \varphi_i^\top(x_1, \dots, x_i)\hat{\theta}_i\right. \\ &\left. + m_i - D^q \beta_{i-1}\right], i = 2, \dots, n-1\end{aligned} \tag{30}$$

and $\beta_1 = -\left[z_1 + k_1 z_1 + \varphi_1^\top(x_1)\hat{\theta}_1 + m_1\right]$ with update laws

$$D^q \hat{\theta}_i = \Gamma_i \varphi_i(x_1, \dots, x_i) z_i \tag{31}$$

and

$$D^q \hat{\delta} = r \sum_{i=1}^{n} \left|z_i\right| \tag{32}$$

Proof. Choose Lyapunov candidate function

$$V_1(z_1, \tilde{\theta}_1, \tilde{\delta}) = \frac{1}{2}z_1^2 + \frac{1}{2}\tilde{\theta}_1^\top \Gamma_1^{-1}\tilde{\theta}_1 + \frac{1}{2r}\tilde{\delta}^2 \tag{33}$$

and

$$V_i = V_{i-1} + \frac{1}{2}z_i^2 + \frac{1}{2}\tilde{\theta}_i^\top \Gamma_i^{-1}\tilde{\theta}_i, i = 2, \dots, n \tag{34}$$

In each step, we have

$$\begin{aligned}D^q V_i \leq -&\sum_{j=1}^{i} k_j z_j^2 + \sum_{j=1}^{i} \tilde{\theta}_j^T \left(\varphi_j z_j - \Gamma_j^{-1} D^q \hat{\theta}_j\right) \\ &+ z_i z_{i+1} + \tilde{\delta}\left(\sum_{j=1}^{i}\left|z_j\right| - \frac{1}{r}D^q\hat{\delta}\right)\end{aligned} \tag{35}$$

According to the control law (29) and update laws (31) and (32), we obtain

$$D^q V_n \leq -\sum_{i=1}^{n} k_i z_i^2 \tag{36}$$

Similar arguments to the proof of Theorem 2 conclude the global asymptotical stability of the closed-loop system. This completes the proof.

IV. EXAMPLES

In this section, two examples of fractional order nonlinear systems are presented to demonstrate our control design. We employ the Radial Basis Function (RBF) to approximate the unknown functions and the Gaussian type functions are used as the basis functions.

Example 1. Consider the fractional order Arneodo's bsystem

$$\begin{aligned} D^q x_1(t) &= x_2(t) \\ D^q x_2(t) &= x_3(t) \\ D^q x_3(t) &= -\varepsilon_1 x_1(t) - \varepsilon_2 x_2(t) - \varepsilon_3 x_3(t) + \varepsilon_4 x_1^3(t) + u \\ y &= x_1 \end{aligned} \tag{37}$$

where $\varepsilon_i (i = 1, \dots, 4)$ are the system parameters and $q \in (0, 1)$. In this example, the unknown function is $f = -\varepsilon_1 x_1 - \varepsilon_2 x_2 - \varepsilon_3 x_3 + \varepsilon_4 x_1^3$. We design the control in three steps:

Step 1. Let $z_1 = x_1$ and define error variable $z_2 = x_2 - \beta_1$, then we have $D^q z_1(t) = z_2 + \beta_1$. Choosing Lyapunov candidate function $V_1 = \frac{1}{2} z_1^2$ and the stabilizing function $\beta_1 = -k_1 z_1$, we have

$$D^q V_1 \leq -k_1 z_1^2 + z_1 z_2 \tag{38}$$

Step 2. Define error variable $z_3 = x_3 - \beta_2$. Choose Lyapunov candidate function $V_2 = V_1 + \frac{1}{2} z_2^2$ and $\beta_2 = -k_2 z_2 - z_1 + D^q \beta_1$, then we have

$$D^q V_2 \leq -k_1 z_1^2 - k_2 z_2^2 + z_2 z_3 \tag{39}$$

Step 3. Choose Lyapunov candidate function $V_3 = V_2 + \frac{1}{2} z_3^2 + \frac{1}{2} \tilde{\theta}^\top \Gamma^{-1} \tilde{\theta} + \frac{1}{2r} \tilde{\delta}^2$ and design the adaptive control

$$u = -\left[k_3 z_3 + z_2 + \varphi^\top \hat{\theta} + sign(z_3)\hat{\delta} - D^q \beta_2\right] \tag{40}$$

and the update laws

$$\begin{aligned} D^q \hat{\theta} &= \Gamma \varphi(x_1, x_2, x_3) z_3 \\ D^q \hat{\delta} &= r \left| z_3 \right| \end{aligned} \tag{41}$$

then we have

$$D^q V_3 \leq -k_1 z_1^2 - k_2 z_2^2 - k_3 z_3^2 \tag{42}$$

Fig. 1 shows the convergence of the Arneodo's system. The system order is $q = 0.97$ and the system parameters are $\varepsilon = [-5.5, 3.5, 1, -1]$ .The initial conditions are $x(0) = [1, 2, 3]^\top$ and $\hat{\theta}(0) = \hat{\delta}(0) = 0$. The gains of update laws are $\Gamma = diag[15]$ and $r = 3$. The control parameters are $k_i = 5$. The number of the RBF neurons is $N = 9$.

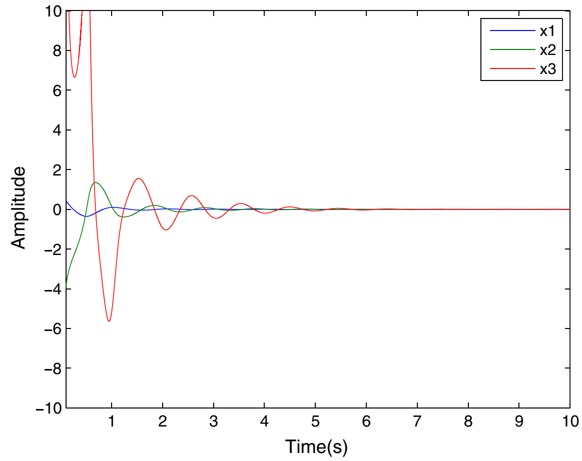

Fig. 1. State convergence in Example 1.

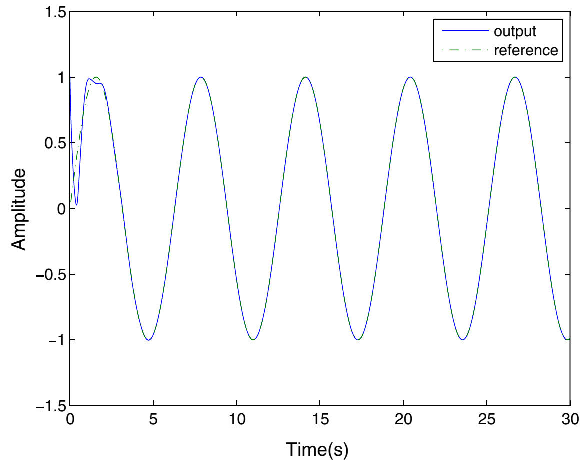

Fig. 2. Signal tracking in Example 1.

To illustrate the approximation of the unknown function f, we design a control, discussed in Remark 5, to achieve the output $y = x_1$ tracking a reference trajectory $y_r = sin(t)$. Figs 2 and 3 show the signal tracking and the function approximation. It can be seen that the unknown function can be well approximated.

Example 2. Consider the following fractional order nonlinear system:

$$\begin{aligned} D^q x_1(t) &= x_2 + f_1\left(x_1\right) \\ D^q x_2(t) &= x_3 + f_2\left(x_1, x_2\right) \\ D^q x_3(t) &= u + f_3\left(x_1, x_2, x_3\right) \\ y &= x_1 \end{aligned} \tag{43}$$

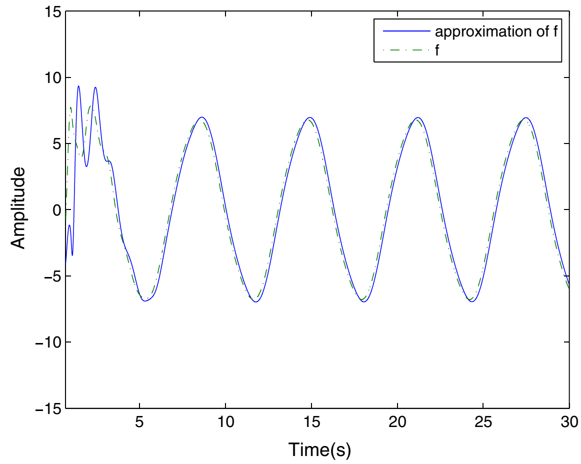

Fig. 3. Approximation of f in Example 1.

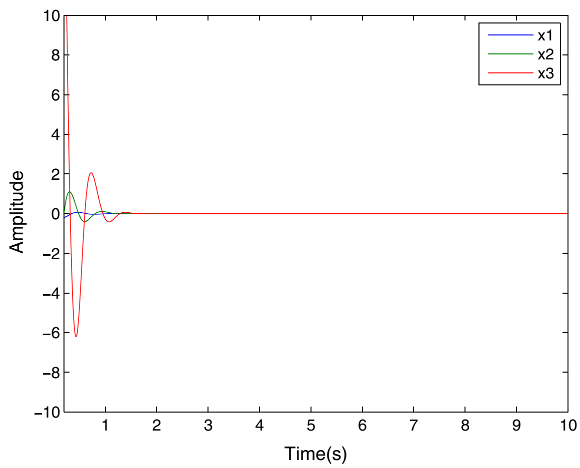

Fig. 4. State convergence in Example 2.

where the unknown functions are

$$\begin{aligned} f_1 &= 2x_1^2\cos(x_1) \\ f_2 &= x_1^2 + x_1x_2 + x_2\sin(x_1) + \frac{x_2}{1+x_1^2} \\ f_3 &= x_1x_3 + x_2\cos(x_3) + \frac{1}{1+x_3^2} \end{aligned} \tag{44}$$

By the design procedure presented in the last section, we have the stabilizing functions

$$\begin{aligned} \beta_1 &= -\left[k_1z_1 + \varphi_1^{\mathrm{T}\hat{\theta}_1} + sign(z_1)\hat{\delta}_1\right] \\ \beta_2 &= -\left[k_2z_2 + z_1 + \varphi_2^{\mathrm{T}\hat{\theta}_2} + sign(z_2)\hat{\delta}_2 - D^q\beta_1\right] \end{aligned} \tag{45}$$

and the control law

$$u = -[k_3z_3 + z_2 + \varphi_3^{\mathrm{T}}(x)\hat{\theta}_3 + sign(z_3)\hat{\delta}_3 - D^q\beta_2] \tag{46}$$

and the update laws

$$\begin{aligned} D^q\hat{\theta}_i &= \Gamma_i\varphi_i(x_1,\ldots,x_i)z_i \\ D^q\hat{\delta}_i &= r_i\left|z_i\right|, i = 1,2,3 \end{aligned} \tag{47}$$

where $z_1 = x_1$, $z_2 = x_2 - \beta_1$ and $z_3 = x_3 - \beta_2$.

Fig. 4 shows the convergence of the system in Example 2. The system order is $q = 0.7$. The initial conditions are $x(0) = [1,2,3]^{\mathrm{T}}$ and $\hat{\theta}(0) = \hat{\delta}(0) = 0$. The gains of update laws are $\Gamma = diag[15]$, $\Gamma = diag[10]$, $\Gamma = diag[12]$ and $r = 2$. The control parameters are $k_i = 9$. The number of the RBF neurons is $N_1 = 9$, $N_2 = 11$ and $N_3 = 10$. Fig. 5 and Fig. 6 show the signal tracking and the function approximation.

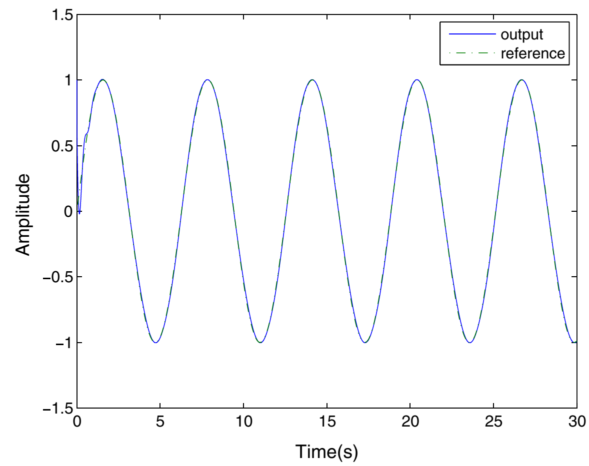

Fig. 5. Signal tracking in Example 2.

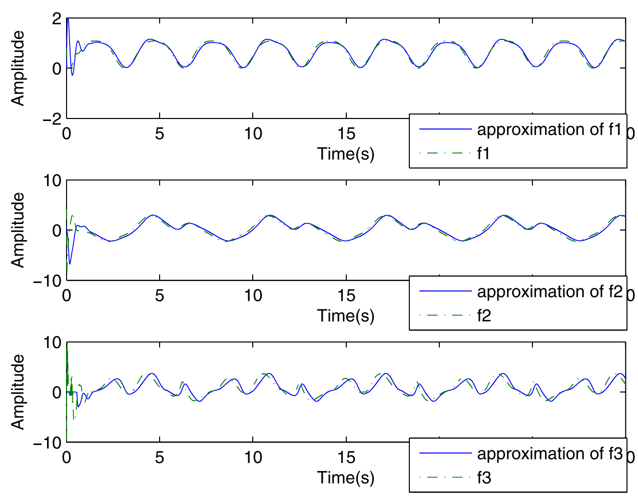

Fig. 6. Approximation of f_i in Example 2.

The simulations in the two examples demonstrate that the proposed control law can stabilize a class of fractional order nonlinear systems with arbitrary uncertainty and the unknown functions can be well approximated.

V. CONCLUSION

This paper deals with stabilization of a class of fractional order nonlinear systems with arbitrary uncertainty. The backstepping design scheme is extended to fractional order systems, and an adaptive control law is proposed with fractional order update laws to achieve a global asymptotical stabilization of the closed-loop system. Examples and simulation results are presented to illustrate the effectiveness of the proposed control.

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